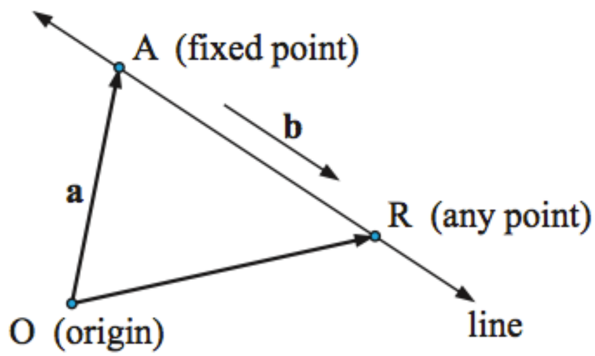
Vector Equation of a Line

A line passes through point A (where A had position vector **a**) and the line is parallel to vector **b**.



R is any point on the line, then $\overrightarrow{AR}$ is parallel to **b**.

$$\overrightarrow{AR} \parallel \mathbf{b}, \quad \overrightarrow{AR} = t\mathbf{b} \text{ for some } t \in \mathbb{R}$$

Using vector addition we can see $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$.

$$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$$

→ The **vector equation** of a line is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{r}$ is the general position vector of a point on the line, $\mathbf{a}$ is a given position vector of a point on the line and $\mathbf{b}$ is a **direction vector** parallel to the line. t is called the parameter.

Find the vector equation through A(1, 5) with direction $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\vec{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Find the vector equation through $(1, -2, 3)$ in the direction $4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

$$\vec{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}$$

Find the vector equation of the line through $A(2, -1, 4)$ and $B(-1, 0, 2)$.

$$\vec{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t\,\cancel{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \text{ or } \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$$

or

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Find the angle between the lines:

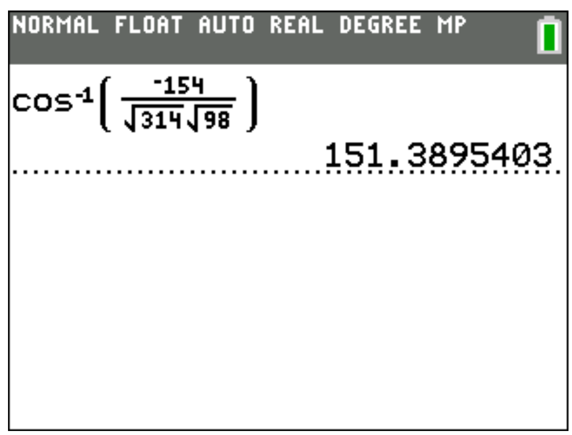
$$L_1: \quad \mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + s \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad L_2: \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}.$$

$$\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \quad \therefore \text{ parallel } \therefore \theta = 0^\circ$$

Find the angle between the lines:

$$L_1: \mathbf{r}_1 = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} + s \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \qquad \text{and} \qquad L_2: \mathbf{r}_2 = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$$

$\theta \approx 151^\circ$



a Find the vector equation of the line through $(1, -1, 3)$ and parallel to the vector $-\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

b Find the vector equation of the line through the points $A(1, 0, -4)$ and $B(-2, 1, 1)$.

c Find the acute angle between these two lines.

a. $\vec{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$

b. $\vec{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$

c. 87.1°

You can think of each part of the vector equation as separate components.

For example consider the following vector equation:

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ where } \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore:

x = 1 - t

y = 5 +3t

This is called parametric form and can useful when solving!

This extends to 3D too!

Another example:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad \text{then } x = 2 + 4t \text{ and } y = 20 - 3t.$$

Does $(3, -2)$ lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

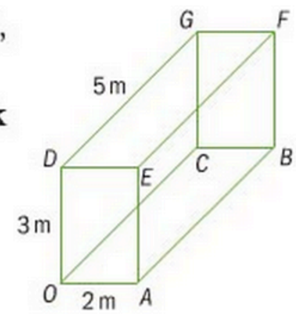
Yes because

$t = 1$

$(k, 4)$ lies on the line $r = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ Find k.

$k = \frac{1}{5}$

10 The figure shows a cuboid in which $OA = 2\,\text{m}$, $OC = 5\,\text{m}$ and $OD = 3\,\text{m}$.
Take O as the origin and unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the direction OA, OC and OD respectively.



a Express these vectors in terms of the unit vectors.
i $\overrightarrow{OF}$ **ii** $\overrightarrow{AG}$

b Calculate the value of
i $|\overrightarrow{OF}|$ **ii** $|\overrightarrow{AG}|$
iii Find the scalar product of $\overrightarrow{OF}$ and $\overrightarrow{AG}$.

c Hence find the angle between the diagonals OF and AG.

Homework

Chapter 12.4

12J: 1 - 11